JOURNAL OF APPROXIMATION THEORY 33, 131–146 (1981)

Uniform Strong Unicity for Rational Approximation

MYRON S. HENRY

Department of Mathematics, Central Michigan University, Mount Pleasant, Michigan 48859

DARRELL SCHMIDT

Department of Mathematical Sciences, Oakland University, Rochester, Michigan 48063

AND

JOHN J. SWETITS

Department of Mathematical Sciences, Old Dominion University, Norfolk, Virginia 23508

Communicated by E. W. Cheney

Received October 14, 1980

Let R_m^n denote the class of rational functions defined on a closed interval I with numerators in the class of polynomials of degree at most n and positive valued denominators in the class of polynomials of degree at most m. If $f \in C(I)$ is normal, the well-known strong unicity theorem asserts that there is a smallest positive constant $\gamma_{n,m}(f)$ such that $\|f - R\| \geqslant \|f - R_f\| + \gamma_{n,m}(f)\|R - R_f\|$ for all $R \in R_m^n$, where R_f is the best uniform approximation to f from R_m^n. In this paper, the dependence of $\gamma_{n,m}(f)$ on f is investigated. Sufficient conditions are given to insure that $\inf_{f \in \Gamma} \gamma_{n,m}(f) > 0$, where Γ is a subset of $C(I)$. Necessity of these conditions is investigated and examples are given to show that known results for R_0^n do not directly extend to R_m^n for $m > 0$.

1. INTRODUCTION

Recently considerable attention has been given to various aspects of strong unicity in best uniform approximation. The focus of the present paper is uniform strong unicity for rational approximation. In particular, let $C(I)$ the set of all continuous, real-valued functions defined on $I = [0, 1]$. If n and m are fixed nonnegative integers, let $\mathscr{S} \subseteq E_{n+m+2}$ consist of the vector $(0,..., 0; 1, 0,..., 0)$ and all vectors $C = (A; B) = (a_0,..., a_n; b_0,..., b_m)$ such that

0021-9045/81/100131-16$02.00/0

Copyright © 1981 by Academic Press, Inc.
All rights of reproduction in any form reserved.

(i) at least one $|a_i| > 0$,

(ii) $P(A, x) = \sum_{i=0}^{n} a_i x^i$ and $Q(B, x) = \sum_{j=0}^{m} b_j x^j$ have no common nonconstant factors,

(iii) $Q(B, x) > 0$ for $x \in I$, and

(iv) $\sum_{j=0}^{m} b_j^2 = 1$.

The approximating space R_m^n is defined to be the set of all rational functions $R(C, x) = P(A, x)/Q(B, x)$ with coefficient vectors $C = (A; B) \in \mathscr{P}$. Let $\mathscr{P}^*$ be the set of all $(A; B) = (a_0, ..., a_n;\ b_0, ..., b_m) \in \mathscr{P}$ such that $a_n \neq 0$ or $b_m \neq 0$. The rational functions $R(C, \cdot)$ with $C \in \mathscr{P}^*$ are the so called *normal* rational functions. For $f \in C(I)$, let $C(f) = (A(f); B(f))$ denote the coefficient vector of $R(C(f), \cdot)$, the best uniform approximation to f from R_m^n. The function f is normal, if its best approximation $R(C(f), \cdot)$ is normal. The well-known strong unicity theorem asserts that if f is normal, then there is a constant $r > 0$ such that

$$\|f - R\| \geqslant \|f - R(C(f), \cdot)\| + r\,\|R - R(C(f), \cdot)\| \tag{1.1}$$

for all $R \in R_m^n$ [4]. Let $\gamma_{n,m}(f)$ denote the largest constant r such that (1.1) holds for every $R \in R_m^n$. In this paper, we study the dependence of $\gamma_{n,m}(f)$ on f. More specifically, if $\Gamma \subseteq C(I)$, when is $\inf_{f \in \Gamma} \gamma_{n,m}(f) > 0$? In the linear approximation setting, this type of question has been analyzed by several authors. In particular, Cline [5] has shown that when $m = 0$, and $n \geqslant 1$, the constant r in (1.1) cannot be chosen to be independent of $f \in C(I)$. In fact, Cline has shown that the pointwise Lipschitz constant for $R(C(f), \cdot)$ (which is bounded above by $2/\gamma_{n,0}(f)$) cannot be selected independent of $f \in C(I)$. In the following theorem due to Henry and Schmidt [9], conditions under which $\gamma_{n,0}(f)$ is bounded away from zero over a subset of $C(I)$ are given.

THEOREM 1.1. *Let $\Gamma \subseteq C(I)$, Γ be compact, and $\Gamma \cap R_0^n = \varnothing$. Then* $\inf_{f \in \Gamma} \gamma_{n,0}(f) > 0$.

Recently Dunham [6] has significantly relaxed the compactness condition in Theorem 1.1 by imposing a "noncoalescence condition."

THEOREM 1.2 (Dunham). *For $\delta > 0$, let F_δ be the set of all $f \in C(I)$ such that there is an alternant $x_0 < x_1 < \cdots < x_{n+1}$ for $f - R(C(f), \cdot)$ such that* $\min_{1 \leqslant i \leqslant n+1} (x_i - x_{i-1}) \geqslant \delta$. *Then* $\inf_{f \in F_\delta} \gamma_{n,0}(f) > 0$.

In addition, Dunham showed that the noncoalescence condition is almost necessary.

THEOREM 1.3 (Dunham). *Let $\{f_k\}$ be a sequence in $C(I)$ such that for each k, $f_k - R(C(f_k), \cdot)$ has precisely one alternant $x_0^k < x_1^k < \cdots < x_{n+1}^k$* and $\lim_{k \to \infty} \min_{1 \leqslant i \leqslant n+1} (x_i^k - x_{i-1}^k) = 0$. *Then* $\lim_{k \to \infty} \gamma_{n,0}(f_k) = 0$.

Actually all of the results above have been obtained in the more general setting of uniform approximation from a Haar subspace.

It is natural to ask whether these results extend to the rational setting (that is, $m > 0$). In [10], it is asserted that a uniform Lipschitz analogue to Theorem 1.1 holds for rational approximation, if each $f \in \Gamma$ is normal; the uniform strong unicity extension will be shown to follow from a more general result in Section 2 of this paper. We also show in Section 2 that Theorem 1.2 does not extend to rational approximation ($m \geqslant 1$) even when the normality condition is imposed. We also show that if the closure of $\{(A(f)/\|f\|; B(f)): f \in \Gamma\}$ is contained in $\mathscr{P}^*$, and if the noncoalescence condition is assumed, then $\gamma_{n,m}(f)$ is bounded away from zero over $\Gamma \subseteq C(I)$. In addition, an example is constructed which shows that even Theorem 1.3 does not extend to R_m^n with $m > 0$. In Section 4, we investigate the necessity of the noncoalescence and closure conditions. It is shown that in certain circumstances neither of these conditions can be omitted if $\inf_{f \in \Gamma} \gamma_{n,m}(f)$ is to be positive.

In order to facilitate the analyses in Section 4, local strong unicity is investigated in Section 3. For $\delta > 0$, define

$$\gamma_{n,m}(f, \delta) = \inf \left\{ \frac{\|f - R\| - \|f - R(C(f), \cdot)\|}{\|R - R(C(f), \cdot)\|} : R \in R_m^n \text{ and } 0 < \|R - R(C(f), \cdot)\| \leqslant \delta \right\}. \quad (1.2)$$

Evidently, $\gamma_{n,m}(f) = \lim_{\delta \to \infty} \gamma_{n,m}(f, \delta)$. *The local strong unicity constant of* f is defined to be

$$\bar{\gamma}_{n,m}(f) = \lim_{\delta \to 0^+} \gamma_{n,m}(f, \delta). \quad (1.3)$$

In Section 3, characterizations of $\bar{\gamma}_{n,m}(f)$ similar to known characterizations of $\gamma_{n,0}(f)$ [1–3, 5, 8, 11] are given. These characterizations will be subsequently used in the necessity considerations of Section 4. Also, in constract to the case $m = 0$, it will be shown that $\gamma_{n,m}(f)$ and $\bar{\gamma}_{n,m}(f)$ need not be equal when $m > 0$. This is one of a number of striking differences in the behavior of polynomial and rational strong unicity constants that will be exhibited in this paper.

2. Uniform Strong Unicity

In this section, we construct an example to show that Theorem 1.2 does not directly extend to the rational setting, and an appropriate extension of this result is then established.

EXAMPLE 1. Let $n=m=1$ and $r_\alpha(x)=\alpha x/(1+\alpha x)$, $\alpha>0$. Define

$$\begin{aligned} h(x) &= 0, && x=0,\\ &= 1, && x=0.25, 0.75,\\ &= -1, && x=0.5, 1 \end{aligned}$$

and let h be linear in between. If $f_\alpha = r_\alpha + h$, then $R(C(f_\alpha), \cdot) = r_\alpha$ where $C(f_\alpha)=(0, \alpha/(1+\alpha^2)^{1/2};\ 1/(1+\alpha^2)^{1/2}, \ \alpha/(1+\alpha^2)^{1/2})$ and each f_α is normal. For each α, the alternant consists of the points 0.25, 0.5, 0.75, 1, and coalescence does not occur. However, for $R(x)=1$, $\|R-r_\alpha\|=1$ and $\lim_{\alpha\to\infty}(\|f_\alpha - R\| - \|f_\alpha - r_\alpha\|)=0$. So

$$\gamma_{1,1}(f_\alpha) \leqslant \frac{\|f_\alpha - R\| - \|f_\alpha - r_\alpha\|}{\|R-r_\alpha\|} \to 0$$

as $\alpha \to 0^+$, and thus $\inf_{\alpha>0} \gamma_{1,1}(f_\alpha)=0$.

THEOREM 2.1. *Let $\Gamma \subseteq C(I)\backslash R_m^n$ satisfy*

(1) *there is a $\delta>0$ such that for each $f\in\Gamma$ there is an alternant $x_0<x_1<\cdots<x_l$ $(l=n+m+1)$ for $f-R(C(f),\cdot)$ such that $\min_{1\leqslant i\leqslant l}(x_i-x_{i-1})\geqslant\delta$ and*

(2) *the closure of $\{(A(f)/\|f\|; B(f)) : f\in\Gamma\}$ in E_{n+m+2} is contained in $\mathscr{P}^*$; then $\inf_{f\in\Gamma}\gamma_{n,m}(f)>0$.*

Proof. Suppose the conclusion is false. Then there is a sequence $\{f_k\}$ in Γ such that $\gamma_{n,m}(f_k)\to 0$ as $k\to\infty$. Since $\gamma_{n,m}(\alpha f)=\gamma_{n,m}(f)$ for $\alpha\neq 0$, we may replace f_k by $f_k/\|f_k\|$, assume that $\|f_k\|=1$, and that

$$\lim_{k\to\infty}\gamma_{n,m}(f_k)=0. \tag{2.1}$$

Also by conditions (1) and (2) we have

(1′) for each k there is an alternant $x_0^k<x_1^k<\cdots<x_l^k$ for $f_k-R(C(f_k),\cdot)$ such that $\min_{1\leqslant i\leqslant l}(x_i^k-x_{i-1}^k)\geqslant\delta$ and

(2′) the closure of $\{(A(f_k); B(f_k)) : k=1,2,\dots\}$ in E_{n+m+2} is contained in $\mathscr{P}^*$.

By (2.1) there is a sequence $R(C_k,\cdot)\in R_m^n$, $C_k=(A_k; B_k)\in\mathscr{P}$, such that

$$\gamma_k := \frac{\|f_k - R(C_k,\cdot)\| - \|f_k - R(C(f_k),\cdot)\|}{\|R(C_k,\cdot)-R(C(f_k),\cdot)\|}\to 0 \tag{2.2}$$

as $k\to\infty$. Since $\|f_k\|=1$, $\|R(C(f_k),\cdot)\|\leqslant 2$ and $\|P(A(f_k),\cdot)\|\leqslant$

$2\|Q(B(f_k),\cdot)\| \leqslant 2(m+1)^{1/2}$. Also, there is an $M>0$ such that $\|R(C_k,\cdot)\| \leqslant M$. Otherwise,

$$\gamma_k \geqslant \frac{\|f_k - R(C_k,\cdot)\| - \|f_k - R(C(f_k),\cdot)\|}{\|f_k - R(C_k,\cdot)\| + \|f_k - R(C(f_k),\cdot)\|} \to 1$$

for a subsequence which is contrary to (2.2). Thus $\|P(A_k,\cdot)\| \leqslant M\|Q(B_k,\cdot)\| \leqslant M(m+1)^{1/2}$. Therefore, the vectors $C(f_k)$ and C_k are bounded independent of k, and we may assume that $C(f_k) \to C$ and $C_k \to \bar{C}$ as $k \to \infty$. Moreover, by (2′) $C \in \mathscr{S}^*$. Thus we have that $P(A(f_k),\cdot) \to P$, $Q(B(f_k),\cdot) \to Q$, $P(A_k,\cdot) \to \bar{P}$, $Q(B_k,\cdot) \to \bar{Q}$ uniformly on I and that $P/Q = R(C,\cdot)$ is a normal rational function in R^n_m. We now renormalize the rational functions by letting

$$N_k = P(A(f_k),\cdot)/(\|P(A(f_k),\cdot)\| + \|Q(B(f_k),\cdot)\|),$$

$$D_k = Q(B(f_k),\cdot)/(\|P(A(f_k),\cdot)\| + \|Q(B(f_k),\cdot)\|),$$

$$N = P/(\|P\| + \|Q\|),$$

$$D = Q/(\|P\| + \|Q\|),$$

$$\bar{N}_k = P(A_k,\cdot)/(\|P(A_k,\cdot)\| + \|Q(B_k,\cdot)\|),$$

$$\bar{D}_k = Q(B_k,\cdot)/(\|P(A_k,\cdot)\| + \|Q(B_k,\cdot)\|),$$

$$\bar{N} = \bar{P}/(\|\bar{P}\| + \|\bar{Q}\|),$$

and

$$\bar{D} = \bar{Q}/(\|\bar{P}\| + \|\bar{Q}\|).$$

We further pass to a subsequence and relabel so that $x_i^k \to x_i$ as $k \to \infty$, $i = 0,\ldots, l$. By (1′), $x_0 < x_1 < \cdots < x_l$. Extracting a subsequence, if necessary, we may assume that $\sigma_i = \operatorname{sgn}(f_k(x_i^k) - R(C(f_k), x_i^k))$, $i = 0,\ldots, l$, where the σ_i alternate in sign and are independent of k. Then for any k and $i = 0,\ldots, l$,

$$\begin{aligned}
&\gamma_k \|R(C_k,\cdot) - R(C(f_k),\cdot)\| \\
&\quad = \|f_k - R(C_k,\cdot)\| - \|f_k - R(C(f_k),\cdot)\| \\
&\quad \geqslant \sigma_i(f_k(x_i^k) - R(C_k, x_i^k)) - \sigma_i(f_k(x_i^k) - R(C(f_k), x_i^k)) \\
&\quad = \sigma_i(R(C(f_k), x_i^k) - R(C_k, x_i^k)) \\
&\quad = \sigma_i(N_k \bar{D}_k - \bar{N}_k D_k)(x_i^k)/D_k(x_i^k)\, \bar{D}_k(x_i^k).
\end{aligned}$$

So $\sigma_i(N_k \bar{D}_k - \bar{N}_k D_k)(x_i^k) \leqslant \gamma_k \|R(C_k,\cdot) - R(C(f_k),\cdot)\|\ \|D_k\|\ \|\bar{D}_k\| \leqslant \gamma_k(M+2)(m+1)$. By (2.2) and the convergences of N_k, D_k, $\bar{N}_k$, and $\bar{D}_k$ to N, D, $\bar{N}$, and $\bar{D}$, respectively, $\sigma_i(N\bar{D} - \bar{N}D)(x_i) \leqslant 0$, $i = 0,\ldots, l$. But this implies that $N\bar{D} - \bar{N}D \equiv 0$. Now Lemma 2 in [4, p. 165] and the normality of N/D imply that $N = \bar{N}$ and $D = \bar{D}$.

Define

$$a_k = \max\{\|p\|: p \in \Pi_{n+m}, \sigma_i p(x_i^k) \leqslant 1, i = 0,..., l\}, \tag{2.3}$$

where Π_{n+m} denotes the space of polynomials of degree $n+m$ or less. It follows from Lemma 3 [9] that $a_k \leqslant a < \infty$ for some constant a independent of k. For each $i = 0,..., l$,

$$\begin{aligned}&\sigma_i(R(C(f_k), x_i^k) - R(C_k, x_i^k))\\&\quad = \sigma_i(f_k(x_i^k) - R(C_k, x_i^k)) - \sigma_i(f_k(x_i^k) - R(C(f_k), x_i^k))\\&\quad \leqslant \|f_k - R(C_k, \cdot)\| - \|f_k - R(C(f_k), \cdot)\| := \Delta_k.\end{aligned}$$

Thus $\sigma_i(N_k\bar{D}_k - \bar{N}_k D_k)(x_i^k) \leqslant \|D_k\| \, \|\bar{D}_k\| \, \Delta_k = (m+1)\, \Delta_k$. By (2.3), $\|N_k\bar{D}_k - \bar{N}_k D_k\| \leqslant a_k(m+1)\, \Delta_k \leqslant a(m+1)\Delta_k$. Thus

$$\|R(C_k, \cdot) - R(C(f_k), \cdot)\| \leqslant a(m+1)\, \|1/D_k\| \, \|1/\bar{D}_k\| \, \Delta_k$$

and

$$\gamma_k \geqslant \{A(m+1)\, \|1/D_k\| \, \|1/\bar{D}_k\|\}^{-1}.$$

Since $D > 0$ on I and $D_k \to D$ and $\bar{D}_k \to D$ uniformly on I as $k \to \infty$, the γ_k are bounded away from zero. This contradicts (2.2) and Theorem 2.1 is proven. ∎

As an application of Theorem 2.1, we show that Theorem 1.1 is valid if $m > 0$ and a normality condition is imposed.

COROLLARY. *Let $\Gamma \subseteq C(I)$, where each $f \in \Gamma$ is normal, Γ is compact, and $\Gamma \cap R_m^n = \varnothing$. Then* $\inf_{f \in \Gamma} \gamma_{n,m}(f) > 0$.

Proof. If the corollary were false, then there would be a sequence $\{f_k\}$ in Γ such that $\gamma_{n,m}(f_k) \to 0$ as $k \to \infty$. Since Γ is compact, we may assume that $f_k \to f$ uniformly as $k \to \infty$, where $f \in \Gamma$ and hence f is normal. Furthermore, we may pass to a subsequence and relabel so that alternant points of $f_k - R(C(f_k), \cdot)$ converge as in the proof of Theorem 1.1 above. Using the argument in the proof of Theorem 3 in [9], we see that in the limit the alternant points of $f_k - R(C(f_k), \cdot)$ do not coalesce, and condition (1) of Theorem 2.1 is satisfied by $\Gamma' = \{f_k : k = 1, 2,...\}$ for some $\delta > 0$. Since f is normal, $C(f_k) \to C(f) \in \mathscr{S}^*$ (see Theorem 1 in [7]). Thus $(A(f_k)/\|f_k\|; B(f_k)) \to (A(f)/\|f\|; B(f)) \in \mathscr{S}^*$ as $k \to \infty$. Hence, condition (2) is satisfied, and Theorem 2.1 provides a contradiction. ∎

Remark. Condition (2) of Theorem 2.1 can be viewed as a strong normality condition. It is of interest to see how it can be violated. Since

$\gamma_{n,m}(\alpha f) = \gamma_{n,m}(f)$, $\alpha \neq 0$, we only consider those $f \in C(I)$ with $\|f\| = 1$. In this case, $|P(A(f), x)| \leqslant 2\,|Q(B(f), x)|$ for $x \in I$. Condition (2) is violated, if there is a sequence f_k such that $P(A(f_k), \cdot) \to P$ and $Q(B(f_k), \cdot) \to Q$ uniformly on I where

$$Q \text{ vanishes at finitely many points in } I \text{ or} \tag{2.4a}$$

$$Q > 0 \text{ on } I \text{ and } P/Q \text{ reduces to } \bar{P}/\bar{Q} \text{ where } \deg \bar{P} < n \text{ and } \deg \bar{Q} < m. \tag{2.4b}$$

In (2.4b), the leading coefficients of $P(A(f_k), \cdot)$ and $Q(B(f_k), \cdot)$ could converge to zero or the limit polynomials P and Q could have common nonconstant factors which do not vanish in I. In Example 1, the failure of condition (2) occurs as a result of (2.4a).

The next example shows that Dunham's necessity result [6, Theorem 1.3] does not hold when $m > 0$. This example also shows that the conditions of Theorem 2.1 are not necessary.

EXAMPLE 2. Let $n = 0$, $m = 1$, and for $k = 2, 3, \ldots$, define $r_k \in R_1^0$ by $r_k(x) = 1/(1 + kx)$ and $h_k \in C(I)$ by

$$\begin{aligned} h_k(x) &= 1, & x &= 0,\ 2/k, \\ &= -1, & x &= 1/k, \\ &= 0, & x &= 1, \end{aligned}$$

and $h_k(x)$ linear in between. Let $f_k = h_k + r_k$. Then r_k is the best approximation to f_k from R_1^0 $(C(f_k) = (1/(1 + k^2)^{1/2};\ 1/(1 + k^2)^{1/2},\ k/(1 + k^2)^{1/2}))$. Each f_k is normal, but $\lim_{k \to \infty}(A(f_k)/\|f_k\|; B(f_k)) = (0; 0, 1) \notin \mathscr{S}^*$ and the alternant $\{0, 1/k, 2/k\}$ for $f_k - r_k$ coalesces to 0. Thus $\{f_k : k = 2, 3, \ldots\}$ fails to satisfy either condition (1) or condition (2) of Theorem 2.1. We show, however, that $\inf_{k \geqslant 2} \gamma_{0,1}(f_k) > 0$.

In this example, let $\|g\|_J = \sup_{x \in J} |g(x)|$. The subscript J will be dropped if $J = [0, 1]$. For fixed k, the change of variable $s = kx/2$ transforms the restrictions of f_k and r_k to $[0, 2/k]$ to $F \in C(I)$ and $R^*(s) = 1/(1 + 2s) \in R_1^0$, respectively. Both F and R^* are independent of k. The rational function R^* is the best approximation to F on I from R_1^0 and F is normal. Applying the strong unicity theorem to F and inverting the change of variable, it follows that there is a constant $\gamma > 0$ independent of k such that

$$\frac{\|f_k - R\|_{[0,2/k]} - \|f_k - r_k\|_{[0,2/k]}}{\|R - r_k\|_{[0,2/k]}} \geqslant \gamma \tag{2.5}$$

for all $R \in R_1^0$ and all $k \geqslant 2$.

Suppose there is a sequence $\{R_k\}$ in R_1^0 such that

$$\frac{\|f_k - R_k\| - \|f_k - r_k\|}{\|R_k - r_k\|} \to 0 \tag{2.6}$$

as $k \to \infty$. Then $\|f_k - R_k\| - \|f_k - r_k\| \to 0$ as $k \to \infty$. Otherwise, (2.6) would imply that a subsequence of $\|R_k - r_k\|$ tends to ∞. This and the boundedness of f_k and r_k would imply that the quotient in (2.6) tends to 1 for the subsequence. Since $\|f_k - r_k\| = 1$, we may assume by (2.6) that $\|f_k - R_k\| = 1 + \varepsilon_k$, where $\varepsilon_k \to 0$ as $k \to \infty$. By (2.5)

$$\begin{aligned}\frac{\varepsilon_k}{\|R_k - r_k\|_{[0,2/k]}} &= \frac{\|f_k - R_k\| - \|f_k - r_k\|}{\|R_k - r_k\|_{[0,2/k]}} \\ &\geqslant \frac{\|f_k - R_k\|_{[0,2/k]} - \|f_k - r_k\|_{[0,2/k]}}{\|R_k - r_k\|_{[0,2/k]}} \geqslant \gamma. \end{aligned} \tag{2.7}$$

Thus

$$\|R_k - r_k\|_{[0,2/k]} \leqslant \varepsilon_k/\gamma. \tag{2.8}$$

Now write $R_k(x) = a_k/(1 + b_k x)$. By (2.8), $|a_k - 1| = |R_k(0) - r_k(0)| \leqslant \varepsilon_k/\gamma$ and

$$\left|\frac{a_k}{1 + b_k/k} - \frac{1}{2}\right| = |R_k(1/k) - r_k(1/k)| \leqslant \varepsilon_k/\gamma.$$

Since $R_k \in R_1^0$, $1 + b_k/k > 0$, and it can easily be shown that

$$\frac{1 - 4\varepsilon_k/\gamma}{1 + 2\varepsilon_k/\gamma} \leqslant b_k/k \leqslant \frac{1 + 4\varepsilon_k/\gamma}{1 - 2\varepsilon_k/\gamma}.$$

Thus $a_k \to 1$ and $b_k/k \to 1$ as $k \to \infty$.

We now consider two cases. First suppose that $\|R_k - r_k\| \leqslant M\,|a_k - 1|$ for infinitely many k, where M is independent of k. Since $|a_k - 1| \leqslant \|R_k - r_k\|_{[0,2/k]}$, (2.7) implies that

$$\frac{\|f_k - R_k\| - \|f_k - r_k\|}{\|R_k - r_k\|} \geqslant \gamma/M$$

for infinitely many k which contradicts (2.6).

In the second case, suppose that

$$\frac{\|R_k - r_k\|}{|a_k - 1|} \to \infty$$

as $k \to \infty$. Consider $\bar{R}_k(x) = R_k(x) + (1-a_k)/(1+b_k x) = 1/(1+b_k x) \in R_1^0$. For k sufficiently large, $\|R_k - r_k\|/|a_k - 1| \geqslant 2$ and so $\|\bar{R}_k - r_k\| \geqslant \|R_k - r_k\| - |a_k - 1| \geqslant \frac{1}{2}\|R_k - r_k\|$. Then

$$\frac{\|f_k - \bar{R}_k\| - \|f_k - r_k\|}{\|\bar{R}_k - r_k\|} \leqslant \frac{\|f_k - R_k\| - \|f_k - r_k\| + |a_k - 1|}{\frac{1}{2}\|R_k - r_k\|}$$

$$= 2\frac{\|f_k - R_k\| - \|f_k - r_k\|}{\|R_k - r_k\|} + 2\frac{|a_k - 1|}{\|R_k - r_k\|} \to 0 \qquad (2.9)$$

as $k \to \infty$. Thus we may replace R_k by $\bar{R}_k$ in (2.6). Now let

$$y_k(x) = \bar{R}_k(x) - r_k(x) = \frac{1}{1+b_k x} - \frac{1}{1+kx}.$$

Since $\lim_{x\to\infty} y_k(x) = 0$, $\|y_k\|_{[0,\infty)} = |y_k(x)|$, where $x = 0$ or $y_k'(x) = 0$ $(x > 0)$. Setting $y_k'(x) = 0$ yields $x = 1/\sqrt{kb_k}$, and since $b_k/k \to 1$ as $k \to \infty$, $1/\sqrt{kb_k} < 2/k$ for k sufficiently large. As a result, $\|\bar{R}_k - r_k\|_{[0,\infty)} = \|\bar{R}_k - r_k\|_{[0,2/k]}$. Thus $\|\bar{R}_k - r_k\| = \|\bar{R}_k - r_k\|_{[0,2/k]}$ and (2.7) now implies that

$$\frac{\|f_k - \bar{R}_k\| - \|f_k - r_k\|}{\|\bar{R}_k - r_k\|} \geqslant \gamma$$

for k sufficiently large which contradicts (2.9). Thus $\inf_{k\geqslant 2} \gamma_{0,1}(f_k) > 0$.

3. Local Strong Unicity Constants

In this section we give a characterization of the local strong unicity constant $\bar{\gamma}_{n,m}(f)$ defined in (1.2) and (1.3). In fact, we show that $\bar{\gamma}_{n,m}(f)$ coincides with the strong unicity constant determined by best approximating $f - R(C(f), \cdot)$ from the linear space

$$\mathscr{M} = \left\{\frac{1}{Q(B(f), \cdot)}(P - R(C(f), \cdot)\, Q) \colon P \in \Pi_n,\ Q \in \Pi_m\right\}. \qquad (3.1)$$

In this section, the function f is fixed and for simplicity we write $R(C(f), \cdot) = R_f = P_f/Q_f$, where $P_f = P(A(f), \cdot)$ and $Q_f = Q(B(f), \cdot)$.

From Lemma 2 in [4, p. 165], $\mathscr{M}$ is a Haar space of dimension $1 + \max\{n + \deg Q_f,\ m + \deg P_f\}$. It follows from the alternation theorem that 0 is the best approximation to $f - R_f$ from $\mathscr{M}$. By the strong unicity theorem for linear approximation, the strong unicity constant for $f - R_f$ is

$$\bar{\bar{\gamma}}_{n,m}(f) := \inf_{\substack{v \in M \\ v \neq 0}} \frac{\|f - R_f - v\| - \|f - R_f\|}{\|v\|} > 0. \tag{3.2}$$

THEOREM 3.1. *If f is normal, then $\bar{\gamma}_{n,m}(f) = \bar{\bar{\gamma}}_{n,m}(f)$.*

Before proving Theorem 3.1, two lemmas are stated. The first asserts that for linear approximation local and global strong unicity constants coincide.

LEMMA 1. *Let X be a normed linear space and V be a subspace of X. For $x \in X$ suppose there exist $v_x \in V$ and $\delta, \gamma > 0$ such that*

$$\|x - v\| \geqslant \|x - v_x\| + \gamma \|v - v_x\| \tag{3.3}$$

for all $v \in V$ with $\|v - v_x\| \leqslant \delta$. Then (3.3) is valid for all $v \in V$.

Proof. Suppose $v \in V$ and $\|v - v_x\| > \delta$. Then $v_x + \delta(v - v_x)/\|v - v_x\| \in V$ and has distance δ from v_x. By (3.3) and the triangle inequality

$$\begin{aligned}(1 - \delta/\|v - v_x\|)\,\|x - v_x\| &+ \delta\,\|x - v\|/\|v - v_x\| \\ &\geqslant \|x - (v_x + \delta(v - v_x)/\|v - v_x\|)\| \\ &\geqslant \|x - v_x\| + \gamma\delta. \end{aligned} \tag{3.4}$$

Inequality (3.3) for the given $v \in V$ now follows directly from (3.4). ∎

If f is normal, then $\mathscr{M}$ has dimension $l = n + m + 1$. It is evident that the l elements $1/Q_f(x)$, $x/Q_f(x), \ldots, x^n/Q_f(x)$, $xR_f(x)/Q_f(x), \ldots, x^m R_f(x)/Q_f(x)$ are linearly independent. Otherwise, we could write $R_f = P/Q$, where $\deg P \leqslant n$, $\deg Q \leqslant m$, and $Q(0) = 0$. If $P(0) = 0$, then a cancellation would occur and R_f would fail to be normal. If $P(0) \neq 0$, then R_f would fail to be continuous at $x = 0$. Thus we may write $\mathscr{M}$ as

$$\mathscr{M} = \{(1/Q_f)(P - R_f Q)\colon P \in \Pi_n,\ Q \in \Pi_m, \text{ and } Q(0) = 0\}. \tag{3.5}$$

LEMMA 2. *Suppose that f is normal.* (i) *Given $\delta > 0$ there is an $\varepsilon > 0$ such that if $P \in \Pi_n$, $Q \in \Pi_m$, $Q(0) = 0$, and $\|(1/Q_f)(P - R_f Q)\| \leqslant \varepsilon$, then*

$$\left\|\frac{P_f + P}{Q_f + Q} - R_f\right\| \leqslant \varepsilon \qquad \textit{and} \qquad \left\|1 - \frac{Q_f}{Q_f + Q}\right\| \leqslant \delta.$$

(ii) *Given* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if* $R = (P_f + P)/(Q_f + Q) \in R_m^n$, $P \in \Pi_n$, $Q \in \Pi_m$, $Q(0) = 0$, *and* $\|R - R_f\| \leqslant \delta$, *then*

$$\left\| 1 - \frac{Q_f + Q}{Q_f} \right\| \leqslant \varepsilon.$$

Lemma 2 follows from the linear independences mentioned above and the fact that $\mathscr{P}^*$ with the Euclidean norm topology and the set of normal rational functions in R_m^n with the uniform norm topology are homeomorphic.

Proof of Theorem 3.1. We first show that $\bar{\bar{\gamma}}_{n,m}(f) \geqslant \bar{\gamma}_{n,m}(f)$. If $\delta > 0$ let $\varepsilon > 0$ be determined as in Lemma 3(i). Now let $v = (1/Q_f)(P - R_f Q)$ be any element of $\mathscr{V}$ satisfying $\|v\| \leqslant \varepsilon$, where $Q_f + Q > 0$ on I. By (1.2)

$$\left\| f - \frac{P_f + P}{Q_f + Q} \right\| \geqslant \|f - R_f\| + \gamma_{n,m}(f, \delta) \left\| \frac{P_f + P}{Q_f + Q} - R_f \right\|. \tag{3.6}$$

But Lemma 2(i) implies that

$$\begin{aligned} \left\| f - \frac{P_f + P}{Q_f + Q} \right\| &= \left\| f - R_f - v + \left(1 - \frac{Q_f}{Q_f + Q}\right) v \right\| \\ &\leqslant \|f - R_f - v\| + \delta \|v\| \end{aligned}$$

and

$$\left\| \frac{P_f + P}{Q_f + Q} - R_f \right\| = \left\| \frac{Q_f}{Q_f + Q} v \right\| \geqslant (1 - \delta) \|v\|.$$

Substituting these into (3.6) yields

$$\|f - R_f - v\| \geqslant \|f - R_f\| + (\gamma_{n,m}(f, \delta)(1 - \delta) - \delta) \|v\| \tag{3.7}$$

for all $v \in \mathscr{V}$ with $\|v\| \leqslant \varepsilon$. By Lemma 1, (3.7) holds for all $v \in \mathscr{V}$, and by (3.2)

$$\bar{\bar{\gamma}}_{n,m}(f) \geqslant \gamma_{n,m}(f, \delta)(1 - \delta) - \delta.$$

Letting $\delta \to 0$, we see that $\bar{\bar{\gamma}}_{n,m}(f) \geqslant \lim_{\delta \to 0^+} \gamma_{n,m}(f, \delta) = \bar{\gamma}_{n,m}(f)$. The proof of the inequality $\bar{\bar{\gamma}}_{n,m}(f) \leqslant \bar{\gamma}_{n,m}(f)$ uses Lemma 3(ii) and is similar to that above. The proof of Theorem 3.1 is now complete. ∎

Since $\mathscr{V}$ is a linear space, Theorem 3.1 now provides a number of characterizations of $\bar{\gamma}_{n,m}(f)$ (see [1–3, 5, 8, 11]). We state two of these characterizations.

The extreme set of $f - R_f$ is defined to be

$$E_{n,m}(f) = \{x \in I \colon |(f - R_f)(x)| = \|f - R_f\|\}$$

and for $x \in E_{n,m}(f)$ let $\sigma(x) = \operatorname{sgn}(f - R_f)(x)$. If f is normal, then

$$\mathscr{M} = \{(1/Q_f^2)(PQ_f - P_fQ): P \in \Pi_n, Q \in \Pi_m\}$$

has dimension $n+m+1$. But $(1/Q_f^2)\,\Pi_{n+m}$ is an $n+m+1$ dimensional subspace of $\mathscr{M}$ and hence

$$\mathscr{M} = \{p/Q_f^2: p \in \Pi_{n+m}\}. \tag{3.8}$$

Theorem 3.2 below follows from Lemma 1 in [3].

THEOREM 3.2. *Let f be normal. Then*

$$\bar{\gamma}_{n,m}(f)^{-1} = \max\{\|p/Q_f^2\|: \quad p \in \Pi_{n+m}, \quad \sigma(x)\,p(x) \leqslant Q_f(x)^2 \quad \textit{for all } x \in E_{n,m}(f)\}.$$

The next result provides a more convenient computation of $\bar{\gamma}_{n,m}(f)$ when it is known that $E_{n,m}(f)$ consists of exactly one alternant and follows from Theorem 5 in [5] and the remark following Theorem 3 in [8].

THEOREM 3.3. *Let f be normal and suppose that $E_{n,m}(f)$ consists of precisely $n+m+2$ points*

$$x_0 < x_1 < \cdots < x_l,$$

where $l = n+m+1$. For $j = 0,..., l$, let $p_j \in \Pi_{n+m}$ satisfy $p_j(x_i) = \sigma(x_i)\, Q_f(x_i)^2$, $i = 0,..., l$, $i \neq j$. Then

$$\bar{\gamma}_{n,m}(f)^{-1} = \max_{0 \leqslant j \leqslant l} \|p_j/Q_f^2\|. \tag{3.9}$$

We conclude this section by noting that in Example 1, $\lim_{\alpha \to 0^+} \bar{\gamma}_{1,1}(f_\alpha) = 1/17$ and $\lim_{\alpha \to 0^+} \gamma_{1,1}(f_\alpha) = 0$. The computation of the first limit can be made using Theorem 3.3. Thus for α sufficiently small, $\gamma_{1,1}(f_\alpha) \neq \bar{\gamma}_{1,1}(f_\alpha)$. Hence when $m > 0$, global and local strong unicity constants need not coincide. This phenomenon is to be contrasted with the case $m = 0$ (see Lemma 1).

4. NECESSITY CONSIDERATIONS

In this section, we study the necessity of the conditions of Theorem 2.1. Although Example 2 indicates that conditions (1) and (2) of Theorem 2.1 are not necessary to ensure uniform strong unicity, a number of general situations are now cited for which the violation of condition (1) or (2) results in strong unicity constants that tend to zero. It will be seen that for certain sequences of functions the global strong unicity constants go to zero while the local strong unicity constants remain bounded away from zero, and in other cases both local and global strong unicity constants tend to zero.

In the remainder of this section, it is assumed that $\{f_k\}$ is a sequence of normal functions contained in $C(I)$ and that $\|f_k\| = 1$ for each k. In view of the remark following the proof of Theorem 2.1, suppose that

$$P(A(f_k), \cdot) \to P \qquad \text{and} \qquad Q(B(f_k), \cdot) \to Q$$

uniformly on I as $k \to \infty$. By restriction (iv) in the parameterization for R^n_m, $Q \not\equiv 0$. From the remark in Section 2, violation of condition (2) of Theorem 2.1 can be expressed in terms of P and Q. Suppose further that each $f_k - R(C(f_k), \cdot)$ has exactly one alternant

$$x_0^k < x_1^k < \cdots < x_l^k,$$

where $l = n + m + 1$. The first result below shows that if condition (1) holds but condition (2) fails according to (2.4a), then uniform strong unicity fails.

THEOREM 4.1. *Suppose that the set* $\{f_k\colon k = 1, 2,...\}$ *satisfies condition* (1) *of Theorem* 2.1 *and* Q *vanishes for some* $z \in I$. *Then* $\lim_{k\to\infty} \bar{\gamma}_{n,m}(f_k) = \lim_{k\to\infty} \gamma_{n,m}(f_k) = 0$.

Proof. We extract a subsequence and relabel so that $x_i^k \to x_i$, $i = 0,..., l$, as $k \to \infty$. By condition (1), $x_0 < x_1 < \cdots < x_l$. Let $\sigma_i = \operatorname{sgn}(f_k(x_i^k) - R(C(f_k), x_i^k))$. By replacing f_k with $-f_k$, if necessary, we may assume that σ_i is independent of k. For convenience, let $x_{-1} = 0$ and $x_{l+1} = 1$. Now select j so that $z \in (x_{j-1}, x_{j+1})$. (If $z = 0$ or 1, close the appropriate end of this interval.) The local strong unicity constant $\bar{\gamma}(f_k)$ is given by (3.9). Select $p_j^k \in \Pi_{n+m}$, where $p_j^k(x_i^k) = \sigma_i Q(B(f_k), x_i^k)^2$, $i = 0,..., l$, $i \neq j$. By Theorem 3.3, $\bar{\gamma}_{n,m}(f_k)^{-1} \geqslant |p_j^k(z)/Q(B(f_k), z)^2|$. Now define $p_j \in \Pi_{n+m}$ by $p_j(x_i) = \sigma_i Q(x_i)^2$, $i = 0,..., l$, $i \neq j$. Then $p_j^k \to p_j$ uniformly on I as $k \to \infty$. Now Q can have at most m zeros, and since $l > m$, $p_j \not\equiv 0$. Since $Q(B(f_k), x)^2 > 0$ on I, p_j^k has $l - 2 = n + m - 1$ zeros in $I \backslash (x_{j-1}^k, x_{j+1}^k)$. (If $z = 0$ or 1, p_j^k has $n + m$ zeros there.) Thus, p_j^k cannot vanish on the strip $x_{j-1}^k < \operatorname{Im}(\zeta) < x_{j+1}^k$ in the complex plane. Using Rouche's theorem it can be seen that $p_j(z) \neq 0$. Thus $\bar{\gamma}_{n,m}(f_k)^{-1} \geqslant |p_j^k(z)/Q(B(f_k), z)^2| \to \infty$ as $k \to \infty$. As we extracted a subsequence, we now have that $\bar{\gamma}_{n,m}(f_{k_v}) \to 0$ as $v \to \infty$ for a subsequence $\{f_{k_v}\}$. However, the above argument shows that every subsequence of $\{f_k\}$ has a subsequence for which the local strong unicity constants tend to zero. Thus $\lim_{k\to\infty} \bar{\gamma}_{n,m}(f_k) = 0$. Since $\gamma_{n,m}(f_k) \leqslant \bar{\gamma}_{n,m}(f_k)$ it follows that $\lim_{k\to\infty} \gamma_{n,m}(f_k) = 0$. ∎

If coalescence occurs and Q does not vanish (that is, condition (2) holds or is violated acording to (2.4b)), then the conclusion of Theorem 4.1 still prevails. This observation constitutes the next theorem.

THEOREM 4.2. *Suppose that* $\min_{1 \leqslant i \leqslant l} (x_i^k - x_{i-1}^k) \to 0$ *as* $k \to \infty$ *and that* Q *does not vanish in* I. *Then* $\lim_{k \to \infty} \bar{\gamma}_{n,m}(f_k) = \lim_{k \to \infty} \gamma_{n,m}(f_k) = 0$.

Proof. Suppose that $x_j^k - x_{j-1}^k \to 0$ as $k \to \infty$, where for simplicity the subscript on j is omitted. Since Q does not vanish on I, there is a constant $\rho > 0$ such that $Q(B(f_k), x) \geqslant \rho$ for all $x \in I$ and all k. For $i \neq j$ and $i \neq j-1$, define p_i^k as in Theorem 3.3 for f_k. Then $\sigma_j p_i^k(x_j^k) = Q(B(f_k), x_j^k)^2 \geqslant \rho^2$ and $\sigma_{j-1} p_i^k(x_{j-1}^k) = Q(B(f_k), x_j^k)^2 \geqslant \rho^2$; thus $|p_i^k(x_j^k) - p_i^k(x_{j-1}^k)| \geqslant 2\rho^2$. Since $x_j^k - x_{j-1}^k \to 0$, the mean value theorem implies that $\|(p_i^k)'\| \to \infty$ as $k \to \infty$. By Markoff's inequality, $\|p_i^k\| \to \infty$ as $k \to \infty$. Since $\|Q(B(f_k), \cdot)\| \leqslant (m+1)^{1/2}$, $\|p_j^k / Q(B(f_k, \cdot)^2\| \to \infty$ as $k \to \infty$, and by Theorem 3.3, $\lim_{k \to \infty} \bar{\gamma}_{n,m}(f_k) = 0$. As in Theorem 4.1, we how have that $\lim_{k \to \infty} \gamma_{n,m}(f_k) = 0$. ∎

Although Example 2 shows that coalescence and the failure of condition (2) as portrayed in (2.4a) can result in uniform strong unicity, the next two theorems indicate that this example is quite sensitive.

THEOREM 4.3. *Suppose that* $x_i^k \to x_i$, $i = 0, \ldots, l$, $x_j = x_{j-1} = z$ *for some* $j = 1, \ldots, l$, *and* $Q(z) \neq 0$. *Then* $\lim_{k \to \infty} \bar{\gamma}_{n,m}(f_k) = \lim_{k \to \infty} \gamma_{n,m}(f_k) = 0$.

The proof of Theorem 4.3 is essentially the same as the proof of Theorem 4.2 and is omitted.

For Example 2, $Q(x) = x$, and we see that $Q(0) = 0$ is necessary to ensure uniform strong unicity. The fact that three alternation points converged to 0 is also necessary. If just two alternant points coalesce, then the conclusion of Theorem 4.1 holds. In fact, if the coalescence is to an interior point of the interval, then coalescence of four or fewer alternant points forces the strong unicity constants to tend to zero.

THEOREM 4.4. *Suppose that* $x_i^k \to x_i$ *as* $k \to \infty$, $i = 0, \ldots, l$, *and that one of the following is satisfied*:

(i) $0 = x_0 = x_1 < x_2 < \cdots < x_l$,

(ii) $x_0 < x_1 < \cdots < x_{l-1} = x_l$, *or*

(iii) *there is a* $j \in \{0, \ldots, l-1\}$ *such that* $x_j = x_{j+1} \leqslant x_{j+2} \leqslant x_{j+3}$, $x_j \in (0, 1)$, *and* $x_0 < \cdots < x_j$ *(if* $j > 0$*) and* $x_{j+3} < \cdots < x_l$ *(if* $j + 3 < l$*).*

Then $\lim_{k \to \infty} \bar{\gamma}_{n,m}(f_k) = \lim_{k \to \infty} \gamma_{n,m}(f_k) = 0$.

The proof of Theorem 4.4 is also omitted, but we note that at the point of coalescence z, Q must vanish at z by Theorem 4.3, if uniform strong unicity is to hold. Then z is a zero of Q^2 of multiplicity at least 2 in cases (i) and (ii) and a zero of multiplicity at least 4 in case (iii). A zero counting argument similar to that given in the proof of Theorem 4.1 shows that p_j can have a zero at z of multiplicity at most 1 in case (i) with $j=0$ and in case (ii) with $j=l$; the multiplicity is at most 3 in case (iii). An application of Theorem 3.3 now yields the result.

We conclude this section with a somewhat more restrictive case in which failure of condition (2) according to (2.4b) forces the global strong unicity constants to tend to zero.

THEOREM 4.5. *Let* $m \leqslant n+1$ *and suppose that* $\inf_{k \geqslant 1} \|f_k - R(C(f_k), \cdot)\| > 0$. *If* $Q > 0$ *on* I *and* P/Q *reduces to* $\bar{P}/\bar{Q}$, *where* $\deg \bar{P} < n$ *and* $\deg \bar{Q} < m$, *then* $\lim_{k \to \infty} \gamma_{n,m}(f_k) = 0$.

Proof. Since $Q > 0$ on I, $R_{f_k} := R(C(f_k), \cdot) \to \bar{P}/\bar{Q}$ uniformly on I as $k \to \infty$. Let $\beta_k = \|f_k - R_{f_k}\|$. If $r_\alpha(x) = 1/(1+\alpha x)$, $\alpha > 0$, then $\bar{P}/\bar{Q} \pm \beta_k r_\alpha \in R_m^n$. Also,

$$\|R_{f_k} - (\bar{P}/\bar{Q} \pm \beta_k r_\alpha)\| \geqslant \beta_k - \|R_{f_k} - \bar{P}/\bar{Q}\| \geqslant \rho$$

for some $\rho > 0$. In addition,

$$\|f_k - (\bar{P}/\bar{Q} \pm \beta_k r_\alpha)\| \leqslant \|f_k - R_{f_k} \mp \beta_k r_\alpha\| + \|R_{f_k} - \bar{P}/\bar{Q}\|.$$

Without loss of generality, assume that $(f_k - R_{f_k})(0) \geqslant 0$. Select a positive sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$. Then there is a $\delta_k > 0$ such that $\beta_k \geqslant (f_k - R_{f_k})(x) \geqslant -\varepsilon_k$ for $0 \leqslant x \leqslant \delta_k$. Also, $\beta_k \geqslant \beta_k r_\alpha(x) > 0$ for $0 \leqslant x \leqslant \delta_k$ and so $|f_k(x) - R_{f_k}(x) - r_\alpha(x)| \leqslant \beta_k + \varepsilon_k$ for $0 \leqslant x \leqslant \delta_k$ and $\alpha > 0$. Since $r_\alpha \to 0$ uniformly on $[\delta_k, 1]$ as $\alpha \to 0$, we may select $\alpha_k > 0$ so that $|\beta_k r_{\alpha_k}(x)| \leqslant \varepsilon_k$ for $x \in [\delta_k, 1]$. Thus $\|f_k - R_{f_k} - \beta_k r_{\alpha_k}\| \leqslant \beta_k + \varepsilon_k$. Hence,

$$\gamma_{n,m}(f_k) \leqslant \frac{\|f_k - (\bar{P}/\bar{Q} + \beta_k r_{\alpha_k})\| - \|f_k - R_{f_k}\|}{\|R_{f_k} - (\bar{P}/\bar{Q} + \beta_k r_{\alpha_k})\|}$$

$$\leqslant \frac{\varepsilon_k + \|R_{f_k} - \bar{P}/\bar{Q}\|}{\rho} \to 0$$

as $k \to \infty$. Thus Theorem 4.5 is proven. ∎

We finally remark that in the case of Theorem 4.5, if coalescence of alternation points does not occur, then the local strong unicity constants $\bar{\gamma}_{n,m}(f_k)$ are bounded away from zero.

REFERENCES

1. M. W. BARTELT, On Lipschitz conditions, strong unicity and a theorem of A. K. Cline, *J. Approx. Theory* **14** (1975), 245–250.
2. M. W. BARTELT AND H. S. MCLAUGHLIN, Characterizations of strong unicity in approximation theory, *J. Approx. Theory* **9** (1973), 255–266.
3. M. W. BARTLET AND D. SCHMIDT, On Poreda's problem on the strong unicity constants, *J. Approx. Theory*, in press.
4. E. W. CHENEY, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
5. A. K. CLINE, Lipschitz conditions on uniform approximation operators, *J. Approx. Theory* **8** (1973), 160–172.
6. C. B. DUNHAM, A uniform constant of strong uniqueness on an interval, *J. Approx. Theory* **28** (1980), 207–211.
7. M. S. HENRY AND J. A. BROWN, Best rational product approximation of functions, *J. Approx. Theory* **9** (1973), 287–294.
8. M. S. HENRY AND J. A. ROULIER, Lipschitz and strong unicity constants for changing dimension, *J. Approx. Theory* **22** (1978), 85–94.
9. M. S. HENRY AND D. SCHMIDT, Continuity theorems for the product approximation operator, *in* "Theory of Approximation with Applications" (A. G. Law and B. N. Sahney, Eds.), pp. 24–42, Academic Press, New York, 1976.
10. M. S. HENRY AND D. SCHMIDT, Continuity theorems for the rational product approximation operator, *J. Approx. Theory* **19** (1977), 135–142.
11. D. SCHMIDT, A characterization of strong unicity constants, *in* "Approximation Theory III" (E. W. Cheney, Ed.), pp. 805–809, Academic Press, New York, 1980.